Linear Programming

Chapter 1-2.2

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Background

$$
\begin{aligned}
x_1 + x_2 - x_3 &= 5 \\
-2x_1 - x_2 + x_3 &= -9 \\
x_1 + 3x_2 - 3x_3 &= 7
\end{aligned}
$$

- System of linear equalities, 2000 years ago
- System of linear inequalities, 18th century
- Linear programming, 20th century

Repetition Gauss Jordan (GJ)

$$\left|\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 5 \\ -2 & -1 & 1 & -9 \\ 1 & 3 & -3 & 7 \end{array}\right|$$

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$$\left|\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 2 & -2 & 2 \end{array}\right|$$

Repetition Gauss Jordan (GJ)

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$$\left|\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 2 & -2 & 2 \end{array}\right|$$

$$\left|\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right|$$

Feasible solution, one redundant row

Repetition Gauss Jordan (GJ)

$$\left|\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 5 \\ -2 & -1 & 1 & -9 \\ 1 & 3 & -3 & \boxed{8} \end{array}\right|$$

Repetition Gauss Jordan (GJ)

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$$\left|\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 2 & -2 & 3 \end{array}\right|$$

Repetition Gauss Jordan (GJ)

$$\left|\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 5 \\ -2 & -1 & 1 & -9 \\ 1 & 3 & -3 & \boxed{8} \end{array}\right|$$

$$\left|\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 2 & -2 & 3 \end{array}\right|$$

$$\left|\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right|$$

Last row shows infeasibility

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Theorem of alternatives for systems of linear equations

Theorem 1. *Excactly one of the following two systems has a solution.*

1. $Ax = b$
2. $\pi = (\pi_1, \ldots, \pi_m)$

$$\sum_{i=1}^{m} \pi_i A_{i.} = \pi A = 0,$$

$$\sum_{i=1}^{m} \pi_i b_i = \pi b = \alpha \neq 0.$$

Memory Matrix in GJ

Original tableau

x	RHS
A	b

Memory Matrix in GJ

Original tableau		Memory matrix
x	RHS	
A	b	I

Memory Matrix in GJ

Original tableau		Memory matrix
x	RHS	
A	b	I

Current tableau		Memory matrix
x	RHS	
$\bar{A}$	$\bar{b}$	$\bar{M}$

Memory Matrix in GJ: Example

BV	PC x_1	x_2	x_3	x_4	x_5	b	Memory matrix					
	$\boxed{1}$	1	1	1	1	−11	1	0	0	0	0	PR
	−1	0	−2	1	0	−3	0	1	0	0	0	
	−2	2	−6	6	2	−34	0	0	1	0	0	
	0	3	−2	−4	−1	2	0	0	0	1	0	
	−2	6	−9	4	2	−40	0	0	0	0	1	

Memory Matrix in GJ: Example

BV	x_1	x_2	x_3	x_4	x_5	b	Memory matrix					
		PC										
x_1	1	1	1	1	1	−11	1	0	0	0	0	
	0	$\boxed{1}$	−1	2	1	−14	1	1	0	0	0	PR
	0	4	−4	8	4	−56	2	0	1	0	0	
	0	3	−2	−4	−1	2	0	0	0	1	0	
	0	8	−7	6	4	−62	2	0	0	0	1	

Memory Matrix in GJ: Example

BV	x_1	x_2	x_3	x_4	x_5	b	Memory matrix					
			PC									
x_1	1	0	2	−1	0	3	0	−1	0	0	0	
x_2	0	1	−1	2	1	−14	1	1	0	0	0	
	0	0	0	0	0	0	−2	−4	1	0	0	RC
	0	0	$\boxed{1}$	−10	−4	44	−3	−3	0	1	0	PR
	0	0	1	−10	−4	50	−6	−8	0	0	1	

Memory Matrix in GJ: Example

BV	x_1	x_2	x_3	x_4	x_5	b	Memory matrix					
x_1	1	0	0	21	10	63	6	5	0	−2	0	
x_2	0	1	0	−8	−4	−54	−2	−2	0	1	0	
	0	0	0	0	0	0	−2	−4	1	0	0	RC
x_3	0	0	1	−10	−4	44	−3	−3	0	1	0	
	0	0	0	0	0	6	−3	−5	0	−1	1	IC

Memory Matrix in GJ: Example

Last row is proof of infeasibility

$$\begin{array}{ccccc|c|ccccc} 0 & 0 & 0 & 0 & 0 & 6 & -3 & -5 & 0 & -1 & 1 \end{array}$$

Where $\pi =$ (-3 -5 0 -1 1)
such that $\pi A = 0$ and $\pi b = 6 \neq 0$

Revised GJ with Explicit Basis Inverse

- $\bar{A}$ is not stored at each iteration
- $\bar{A}$ can be computed: columns $\bar{A}_{.j} = \bar{M} A_{.j}$ and rows $\bar{A}_{i.} = \bar{M}_{i.} A$

Revised GJ with Explicit Basis Inverse

- $\bar{A}$ is not stored at each iteration
- $\bar{A}$ can be computed: columns $\bar{A}_{.j} = \bar{M} A_{.j}$ and rows $\bar{A}_{i.} = \bar{M}_{i.} A$
- Used in computer implementations to save memory
- Similar to Dantzigs revised simplex method
- Memory matrix referred to as basis inverse, denoted B^{-1}

Revised GJ with Explicit Basis Inverse

Method

1. Select pivot row i
2. Compute row i: $\bar{A}_{i.}$
3. If $\bar{A}_i \neq 0$, select nonzero pivot element j
   If $\bar{A}_i = 0$, either row is redundant, go to 1 or problem is infeasible, method finishes.
4. Compute column j: $\bar{A}_{j.}$ and perform pivot step
5. Stop when pivot step has been done for all rows

Revised GJ: Example

Original system						BV	Memory matrix							
x_1	x_2	x_3	x_4	x_5	b	BV	Inverse tableau B^{-1}					PC x_1	
1	1	1	1	1	−11		1	0	0	0	0	$\boxed{1}$	PR
−1	0	−2	1	0	−3		0	1	0	0	0	−1	
−2	2	−6	6	2	−34		0	0	1	0	0	−2	
0	3	−2	−4	−1	2		0	0	0	1	0	0	
−2	6	−9	4	2	−40		0	0	0	0	1	−2	

Revised GJ: Example

Original system		BV	Memory matrix						
x_1 x_2 x_3 x_4 x_5	b	BV	Inverse tableau B^{-1}					PC x_2	
Now PR = Row 2.	-11	x_1	1	0	0	0	0	1	
$\bar{A}_{2.} = (1, 1, 0, 0, 0)A$	-14		1	1	0	0	0	$\boxed{1}$	PR
$= (0, 1, -1, 2, 1)$.	-56		2	0	1	0	0	4	
x_2 selected EV.	2		0	0	0	1	0	3	
PC $= B^{-1}A_{.2}$ entered.	-62		2	0	0	0	1	8	

Revised GJ: Example

Original system							Memory matrix						
x_1	x_2	x_3	x_4	x_5	b	BV	Inverse tableau B^{-1}					PC x_3	
PR = Row 3. $\bar{A}_{3.} = (-2, -4, 1, 0, 0)A = 0$. $\bar{b}_3 = 0$. RC.													
PR = Row 4. $\bar{A}_{4.} = (-3, -3, 0, 1, 0)A = (0, 0, 1, -10, -5)$. EV = x_3. PC = $B^{-1}A_{.3}$ entered.					3	x_1	0	−1	0	0	0	2	
					−14	x_2	1	1	0	0	0	−1	
					0		−2	−4	1	0	0	0	
					44		−3	−3	0	1	0	$\boxed{1}$	PR
					50		−6	−8	0	0	1	1	

Revised GJ: Example

Original system							Memory matrix					
x_1	x_2	x_3	x_4	x_5	b	BV	Inverse tableau B^{-1}					
PR = Row 5. $\bar{A}_{5.} = (-3, -5,$					63	x_1	6	5	0	-2	0	
$0, -3, 1)A = 0$. $\bar{b}_5 = 6$. IC.					-54	x_2	-2	-2	0	1	0	
Infeasible.					0		-2	-4	1	0	0	
					44	x_3	-3	-3	0	1	0	
					6		-3	-5	0	-1	1	

Systems of Linear Inequalities

Theorem 1.2. *Consider the system of linear inequalities*

$$Ax \geq b, \tag{1.4}$$

where $A = (a_{ij})$ *is an* $m \times n$ *matrix and* $b = (b_i) \in R^m$. *So, the constraints in the system are* $A_{i.}x \geq b_i$, $i \in \{1, \ldots, m\}$. *If this system has a feasible solution, then there exists a subset* $\mathbf{P} = \{p_1, \ldots, p_s\} \subset \{1, \ldots, m\}$ *such that every solution of the system of equations*

$$A_{i.}x = b_i, \quad i \in \mathbf{P},$$

is also a feasible solution of the original system of linear inequalities (1.4).

Systems of Linear Inequalities

Start with x^0 and P_0 indices of active constraints.

- If $P_0 = \emptyset$: Select a constraint i and a point $\bar{x}$ on the constraint. If $\bar{x}$ is infeasible. Find maximum λ such that $x^1 = x^0 + \lambda(\bar{x} - x^0)$ is feasible.

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In iteration r

1. If x^r is unique solution to system, terminate.

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1. If x^r is unique solution to system, terminate.
2. Let $\{y\}$ be basis for $\{A_{i.}y = 0; i \in P_r\}$

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1. If x^r is unique solution to system, terminate.
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3. If $\{A_{i.}y = 0; \forall y, i\}$ terminate

Systems of Linear Inequalities

Start with x^0 and P_0 indices of active constraints.

- If $P_0 = \emptyset$: Select a constraint i and a point $\bar{x}$ on the constraint. If $\bar{x}$ is infeasible. Find maximum λ such that $x^1 = x^0 + \lambda(\bar{x} - x^0)$ is feasible.

In iteration r

1. If x^r is unique solution to system, terminate.
2. Let $\{y\}$ be basis for $\{A_{i.}y = 0; i \in P_r\}$
3. If $\{A_{i.}y = 0; \forall y, i\}$ terminate
4. Otherwise, take $\bar{y}$ such that $A_{i.}\bar{y} < 0$ for some i. Find maximum λ such that $x^{r+1} = x^r + \lambda(\bar{y} - x^r)$ is feasible.

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Convex sets

Definition 1. *A set K is convex if for*
$x, y \in K, 0 \leq \alpha \leq 1$,
then $z = \alpha x + (1 - \alpha)y \in K$

Convex functions

Jensen's inequality
Let $0 \leq \alpha \leq 1$ and $y^1, y^2 \in \Gamma$ where Γ is a convex set.

Definition 2. *A function* $g(y)$ *is convex if*
$g(\alpha y^1 + (1-\alpha)y^2) \leq \alpha g(y^1) + (1-\alpha)g(y^2)$

Concave functions

Let $0 \leq \alpha \leq 1$ and $y^1, y^2 \in \Gamma$ where Γ is a convex set.

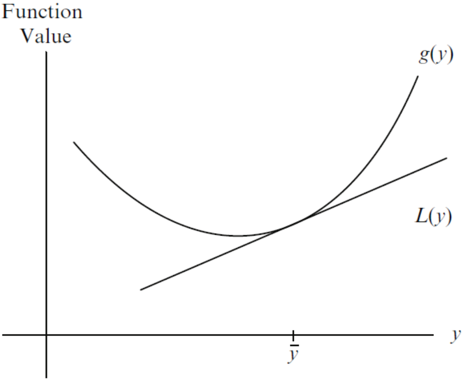
Definition 3. *A function $h(y)$ is concave if*
$h(\alpha y^1 + (1-\alpha)y^2) \geq \alpha h(y^1) + (1-\alpha)h(y^2)$

Gradient support inequality

Theorem 2. *Let $g(y)$ be a real-valued differentiable real-valued function defined on $\mathbb{R}^n$.*

Then $g(y)$ is convex iff $g(y) \geq g(\bar{y}) + \nabla g(\bar{y})(y - \bar{y})$

Gradient support inequality



Differentiable function

Theorem 3. *Let $g(y)$ be a real-valued differentiable real-valued function defined on $\mathbb{R}^n$.*

Then $g(y)$ is convex iff $(\nabla g(y^2) - \nabla g(y^1))(y^2 - y^1) \geq 0$

Twice differentiable function

Theorem 4. *Let $g(y)$ be a twice continously differentiable real-valued function defined on $\mathbb{R}^n$.*

1. *$g(y)$ is convex iff the Hessian $H(g(y)) = (\frac{\partial^2 g(y)}{\partial y_i \partial y_j})$ is positive semi-definite.*
2. *$g(y)$ is concave iff the Hessian is negative semi-definite.*

Twice differentiable function: In practice

Using Hessian to check convexity

- Hard in the general case
- Easy for quadratic functions $f(x) = xDx + cx + c_0$
  Hessian equals $\frac{D+D^T}{2}$ and is constant

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